

ON FANO VARIETIES WITH TORUS ACTION OF COMPLEXITY ONE

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ABSTRACT. In this work we provide effective bounds and classification results for rational \mathbb{Q} -factorial Fano varieties with a complexity one torus action and Picard number one depending on the invariants dimension and Picard index. This complements earlier work, where the case of free divisor class group of rank one was treated.

1. STATEMENT OF THE RESULTS

We consider rational \mathbb{Q} -factorial Fano varieties X over an algebraically closed field \mathbb{K} of characteristic zero that come with an effective action of a torus T of complexity one, i.e. $\dim X - \dim T = 1$; by Fano varieties we mean normal projective varieties with ample anticanonical divisor $-K_X$. We continue the work of [6] where classification results for the case $\mathrm{Cl}(X) = \mathbb{Z}$ were given. Here we just require Picard number one, i.e. we allow torsion in the divisor class group. A first step is Theorem 3.3 where we provide effective bounds for the number of deformation types of Fano varieties X as above with fixed dimension d and Picard index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. As a consequence we obtain the following asymptotical statements about the number $\delta(d, \mu)$ of different deformation types of \mathbb{Q} -factorial d -dimensional Fano varieties with a complexity one torus action, Picard number one and Picard index μ :

Theorem 1.1. *For fixed $d \in \mathbb{Z}_{>0}$, the number $\delta(d, \mu)$ is asymptotically bounded by μ^{μ^2} , and for fixed $\mu \in \mathbb{Z}_{>0}$, it is asymptotically bounded by d^{Ad} with a constant A depending only on μ .*

We turn to the classification. Our approach uses the Cox ring $\mathcal{R}(X)$. Note that every Fano variety is uniquely determined by its Cox ring (as a $\mathrm{Cl}(X)$ -graded ring). In the subsequent theorems we list the cases where $\mathrm{Cl}(X)$ has non trivial torsion; for the results in case of $\mathrm{Cl}(X) = \mathbb{Z}$ we refer to [6]. The Cox rings are described in terms of generators and relations and we specify the $\mathrm{Cl}(X)$ -grading by giving the degrees of the generators. Additionally we list the degree of the Fano varieties $d_X := (-K_X)^d$ and the Gorenstein index $\iota(X)$, i.e. the smallest positive integer such that $\iota(X) \cdot K_X$ is Cartier.

Theorem 1.2. *Let X be a non-toric Fano surface with an effective \mathbb{K}^* -action, Picard number one, non trivial torsion in the class group and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \leq 6$. Then its Cox ring is precisely one of the following.*

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$$

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
1	$\mathbb{K}[T_1, \dots, T_4] / \langle T_1 T_2^3 + T_3^4 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	1	1

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 3$$

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
2	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1 T_2^2 + T_3^3 + T_4^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{1}{1} & \frac{1}{2} & \frac{1}{0} \end{pmatrix}$	1	1

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 4$$

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
3	$\mathbb{K}[T_1, T_2, T_3, S_1]/\langle T_1^2 + T_2^2 + T_3^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{1}{1} & \frac{1}{0} & \frac{1}{0} \\ \frac{0}{0} & \frac{1}{1} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}$	2	1
4	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1 T_2 + T_3^2 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{1}{3} & \frac{1}{2} & \frac{1}{0} \end{pmatrix}$	2	1
5	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^2 T_2 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{2}{0} & \frac{2}{1} & \frac{1}{0} \end{pmatrix}$	2	1
6	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1 T_2^2 + T_3^6 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{2}{0} & \frac{2}{1} & \frac{3}{0} & \frac{1}{1} \end{pmatrix}$	1	2
7	$\mathbb{K}[T_1, \dots, T_5]/\langle \frac{T_1 T_2 + T_3^2 + T_4^2}{\lambda T_3^2 + T_4^2 + T_5^2} \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{0} & \frac{1}{0} \\ \frac{1}{0} & \frac{1}{0} & \frac{1}{1} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}$	1	1

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 6$$

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
8	$\mathbb{K}[T_1, T_2, T_3, S_1]/\langle T_1^3 + T_2^3 + T_3^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} \frac{2}{1} & \frac{2}{1} & \frac{3}{0} & \frac{1}{0} \end{pmatrix}$	2/3	3
9	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1 T_2 + T_3^3 + T_4^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{2}{2} & \frac{1}{1} & \frac{1}{0} \end{pmatrix}$	2	1
10	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1 T_2 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{3}{1} & \frac{1}{1} & \frac{2}{1} & \frac{1}{0} \end{pmatrix}$	3	1
11	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1 T_2^5 + T_3^2 + T_4^8 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{3}{1} & \frac{1}{1} & \frac{4}{1} & \frac{1}{0} \end{pmatrix}$	1/3	3

where the parameter λ occurring in the second relation of surface number 7 can be any element of $\mathbb{K}^* \setminus \{1\}$.

Remark 1.3. The Gorenstein surfaces are well known to have ADE-singularities which are in particular canonical. Consequently the surfaces of number 1 to 5, 7, 9 and 10 are canonical. Further in [9] all log-terminal Del Pezzo \mathbb{K}^* -surfaces of Gorenstein index up to three are classified. Comparing the surfaces listed in [9, Theorems 4.9, 4.10] with the table above shows that number 11 is not log-terminal. The resolution of this surface can be explicitly computed by using the method of toric ambient modification as demonstrated in [4, Examples 3.20, 3.21].

Theorem 1.4. *Let X be a three-dimensional non-toric Fano variety with an effective two torus action, Picard number one, non trivial torsion in the class group and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
1	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1 T_2 + T_3^2 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{1} & \frac{1}{1} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}$	27	1
2	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1 T_2^3 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{2}{1} & \frac{1}{0} & \frac{1}{1} \end{pmatrix}$	8	2
3	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1 T_2^3 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{2}{1} & \frac{1}{1} & \frac{1}{0} \end{pmatrix}$	8	1
4	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1 T_2^3 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{2}{1} & \frac{1}{1} & \frac{1}{1} \end{pmatrix}$	8	2
5	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1 T_2^5 + T_3^2 + T_4^6 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{3}{0} & \frac{1}{1} & \frac{1}{1} \end{pmatrix}$	1	1

[illegible]

Let X be a normal complete rational variety coming with a complexity one torus action of T . Consider the T -invariant open subset X_0 consisting of all points $x \in X$

having finite isotropy group. According to [8, Cor. 3] there is a geometric quotient $q: X_0 \rightarrow X_0/T$ such that X_0/T is irreducible and normal but possibly not separated. The property of the orbit space X_0/T being separated is reflected in the Cox ring relations by the condition that each monomial depends on only one variable, e.g. surface number 3 in Theorem 1.2; see [7, Theorem 1.2]. For such varieties we have the following general finiteness statement:

Theorem 1.5. *The number of d -dimensional normal complete rational varieties of Picard number one with a complexity one torus action of T and Picard index μ such that X_0/T is separated is finite.*

2. DESCRIPTION OF THE COX RING

We briefly recall from [5] a construction of \mathbb{Q} -factorial normal projective varieties with a complexity one torus action and Picard number one; the details are given in [5, Prop. 2.4].

Construction 2.1. For $r \geq 1$, consider a sequence $A = (a_0, \dots, a_r)$ of pairwise linearly independent vectors in \mathbb{K}^2 , a sequence $\mathbf{n} = (n_0, \dots, n_r)$ of positive integers, a non-negative integer m and a family $L = (l_{ij})$ of positive integers, where $0 \leq i \leq r$ and $1 \leq j \leq n_i$. Set

$$R(A, \mathbf{n}, L, m) := \mathbb{K}[T_{ij}, S_k] / \langle g_0, \dots, g_{r-2} \rangle.$$

where the T_{ij} are indexed by $0 \leq i \leq r$, $1 \leq j \leq n_i$, the S_k by $1 \leq k \leq m$ and the relations g_i are defined as follows: Set $T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ and

$$g_i := \det \begin{pmatrix} a_i & a_{i+1} & a_{i+2} \\ T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \end{pmatrix}.$$

Define $n := n_0 + \dots + n_r$ and let $K := \mathbb{Z} \oplus K^t$ be an abelian group with torsion part K^t . Suppose that $R(A, \mathbf{n}, L, m)$ is K -graded via

$$\deg T_{ij} = w_{ij} \in K, \quad \deg S_k = u_k \in K,$$

such that these degrees generate a pointed cone in $K_{\mathbb{Q}} = K \otimes \mathbb{Q}$ and any $n + m - 1$ of them generate K as a group. The K -grading defines a diagonal action of $H := \text{Spec } \mathbb{K}[K]$ on \mathbb{K}^{n+m} . By construction

$$\overline{X} := V(g_i; 0 \leq i \leq r-2) = \text{Spec } R(A, \mathbf{n}, L, m)$$

is invariant under this H -action. The open set $\widehat{X} := \overline{X} \setminus \{0\}$ allows a geometric quotient $p: \widehat{X} \rightarrow X$. The quotient space $X := \widehat{X} // H$ is a \mathbb{Q} -factorial normal projective variety of dimension

$$\dim(X) = n + m - r.$$

It has divisor class group $\text{Cl}(X) = K$, Cox ring $\mathcal{R}(X) = R(A, \mathbf{n}, L, m)$ and comes with a complexity one torus action.

According to [5, Theorem 1.5] every \mathbb{Q} -factorial normal complete rational variety X with a complexity one torus action and Picard number one has a Cox ring $\mathcal{R}(X)$ which is isomorphic as a graded ring to some K -graded algebra $R(A, \mathbf{n}, L, m)$ with $K \cong \text{Cl}(X)$.

We collect some geometric properties of the varieties X just constructed. Every element $w \in K = \mathbb{Z} \oplus K^t$ can be written as $w = w^0 + w^t$ where $w^0 \in \mathbb{Z}$ and $w^t \in K^t$. Further every $\overline{x} = (\overline{x}_{ij}, \overline{x}_k) \in \widehat{X} \subseteq \mathbb{K}^{n+m}$ defines a point $x \in X$ by $x := p(\overline{x})$; the points $\overline{x} \in \widehat{X}$ are called Cox coordinates of x . We denote the set of all weights corresponding to a non-zero coordinate of \overline{x} by

$$W_{\overline{x}} := \{w_{ij}; \overline{x}_{ij} \neq 0\} \cup \{u_k; \overline{x}_k \neq 0\}.$$

Proposition 2.2. *Let X be a \mathbb{Q} -factorial complete normal variety with complexity one torus action and Picard number one as constructed in 2.1 and set $\gamma_i := \deg(g_i)$, $0 \leq i \leq r$. Then the following statements hold:*

- (i) *For any $\bar{x} \in \hat{X}$, the local divisor class group $\text{Cl}(X, x)$ of $x := p(\bar{x})$ is finite and $\gcd(w^0; w \in W_{\bar{x}})$ is always a divisor of the group order.*
- (ii) *The Picard group $\text{Pic}(X)$ is free and the Picard index is given by*

$$[\text{Cl}(X) : \text{Pic}(X)] = \text{lcm}_{x \in X}(\gcd(w^0; w \in W_{\bar{x}})) \cdot |\text{Cl}(X)^t|.$$

- (iii) *For the anticanonical class $-K_X \in \text{Cl}(X)$ and its self intersection number $d_X := (-K_X)^d$ one has*

$$-K_X = \sum_{i=0}^r \sum_{j=1}^{n_i} w_{ij} + \sum_{k=1}^m u_k - \sum_{i=0}^{r-2} \gamma_i,$$

$$d_X = \left(\sum_{i=0}^r \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{k=1}^m u_k^0 - \sum_{i=0}^{r-2} \gamma_i^0 \right)^d \frac{\gamma_0^0 \cdots \gamma_{r-2}^0}{\prod_{i=0}^{r-2} \prod_{j=1}^{n_i} w_{ij}^0 \prod_{k=1}^m u_k^0 \cdot |\text{Cl}(X)^t|}.$$

- (iv) *The variety X is Fano if and only if the following inequality holds:*

$$\sum_{i=0}^{r-2} \deg(g_i)^0 < \sum_{i=0}^r \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{k=1}^m u_k^0.$$

Proof. Let $\bar{x}(i, j)$ resp. $\bar{x}(k)$ be a point in \hat{X} having the ij -th resp. $(n+k)$ -th entry one and all others zero. With $\hat{Z} := \mathbb{K}^{n+m} \setminus \{0\}$ we obtain a commutative diagram

$$\begin{array}{ccc} \hat{X} & \hookrightarrow & \hat{Z} \\ \parallel H \downarrow & & \downarrow \parallel H \\ X & \hookrightarrow & Z \end{array}$$

where the induced map embeds X into a toric variety Z such that $\text{Cl}(X) \cong \text{Cl}(Z)$ and $\text{Pic}(X) \cong \text{Pic}(Z)$ holds; see [1, Cor. III.3.1.7]. By choice $\bar{x}(i, j)$ resp. $\bar{x}(k)$ is a toric fixed point. Consequently, the Picard group $\text{Pic}(Z)$, and also $\text{Pic}(X)$, is free [2, Theorem VII. 2.16]. According to [3, Cor. 4.9] we obtain

$$\text{Pic}(X) = \bigcap_{\bar{x} \in \hat{X}} \langle w; w \in W_{\bar{x}} \rangle = \bigcap_{\bar{x} \in \hat{X}} \langle w^0; w \in W_{\bar{x}} \rangle,$$

where the last equality follows from the fact that $\text{Pic}(X)$ is free. This proves assertions (i) and (ii). The remaining statements are special cases of [3, Prop. 4.15 and Cor. 4.16]. The self intersection number can be easily computed by using toric intersection theory in the ambient toric variety; compare [1, Constr. III 3.3.4]. \square

Corollary 2.3. *Let X be a \mathbb{Q} -factorial normal complete variety with complexity one torus action and Picard number one. If X is locally factorial, then the divisor class group $\text{Cl}(X)$ is free.*

Corollary 2.4. *Let X be a \mathbb{Q} -factorial complete normal variety with complexity one torus action and Picard number one. Then $|\text{Cl}(X)^t|$ divides $\mu = [\text{Cl}(X) : \text{Pic}(X)]$ and in particular we get $|\text{Cl}(X)^t| \leq \mu$.*

The following example shows that one can use Proposition 2.2(iv) to create series of Fano varieties by altering the torsion part of the divisor class group $\text{Cl}(X)$:

Example 2.5. Set $l_{01} = 7$, $l_{02} = 1$, $l_{11} = 5$ and $l_{21} = 2$ as well as $w_{01}^0 = 1$, $w_{02}^0 = 3$, $w_{11}^0 = 2$ and $w_{21}^0 = 5$. According to Construction 2.1 this data defines one single Cox ring relation of the form $g_0 = T_{01}^7 T_{02} + T_{11}^5 + T_{21}^2$. Since we have

$$w_{01}^0 + w_{02}^0 + w_{11}^0 + w_{21}^0 = 11 > 10 = \deg(g_0)^0,$$

one can use this data to create Cox rings of Fano varieties. We provide some possible $\text{Cl}(X)$ -gradings, given by the grading matrices Q_i , defining Del Pezzo \mathbb{K}^* -surfaces that hold the fixed data but vary in the torsion part of the class group $\text{Cl}(X)^t$:

$$\begin{aligned} Q_1 &= \begin{pmatrix} 1 & 3 & 2 & 5 \end{pmatrix}, & \text{Cl}(X_1) &= \mathbb{Z}; \\ Q_2 &= \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \end{pmatrix}, & \text{Cl}(X_2) &= \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}; \\ Q_3 &= \begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & 1 & 3 & 3 \end{pmatrix}, & \text{Cl}(X_3) &= \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}; \\ Q_4 &= \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 9 & 6 \end{pmatrix}, & \text{Cl}(X_4) &= \mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z}; \\ Q_5 &= \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 3 & 11 & 8 \end{pmatrix}, & \text{Cl}(X_5) &= \mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z}; \\ Q_6 &= \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 15 & 12 \end{pmatrix}, & \text{Cl}(X_6) &= \mathbb{Z} \oplus \mathbb{Z}/17\mathbb{Z}. \end{aligned}$$

Note that in this situation not every group of the form $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, $k \in \mathbb{N}_{>0}$, can be realized as divisor class group.

In Example 2.5 the numbers $\ell_i := \gcd(l_{i1}, \dots, l_{in_i})$ are pairwise coprime, namely $\ell_0 = 1$, $\ell_1 = 2$ and $\ell_2 = 5$. This allows the case $\text{Cl}(X_1) = \mathbb{Z}$; see [6, Theorem 1.9]. If the numbers ℓ_i are not pairwise coprime, then there is always non trivial torsion in the divisor class group as the following Lemma shows.

Lemma 2.6. *Set $\ell_i := \gcd(l_{i1}, \dots, l_{in_i})$. Then all numbers $\gcd(\ell_i, \ell_j)$, where $0 \leq i \neq j \leq r$, divide $|\text{Cl}(X)^t|$ and the Picard index μ . In particular this holds for $\text{lcm}_{j \neq i}(\gcd(\ell_i, \ell_j))$.*

Proof. According to [5, Theorem 1.5] the divisor class group $\text{Cl}(X)$ is isomorphic to $\mathbb{Z}^{n+m}/\text{im}(P^*)$ where P^* is dual to $P: \mathbb{Z}^{n+m} \rightarrow \mathbb{Z}^{n+m-1}$ given by a matrix of the form

$$P = \begin{pmatrix} -l_0 & l_1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ -l_0 & 0 & \dots & l_r & 0 \\ d_0 & d_1 & \dots & d_r & d' \end{pmatrix},$$

with $l_i = (l_{i0}, \dots, l_{in_i})$ and some integral block matrices d_i and d' . Consequently $|\text{Cl}(X)^t|$ is the product of all elementary divisors of P which implies that $\gcd(\ell_0, \ell_j)$ divides $|\text{Cl}(X)^t|$. By an elementary row transformation we obtain the analogous result for $\gcd(\ell_i, \ell_j)$ where $0 \leq i, j \leq r$, $i \neq j$. Since $|\text{Cl}(X)^t|$ divides the Picard index μ , the assertion follows. \square

Remark 2.7. One can even prove that $\text{lcm}_{0 \leq j \leq r}(\prod_{i \neq j} \gcd(\ell_i, \ell_j))$ divides $|\text{Cl}(X)^t|$ (see for example surface number 3 in Theorem 1.2).

3. EFFECTIVE BOUNDS

First we consider the case $n_0 = \dots = n_r = 1$, that means that each relation g_i of the Cox ring $\mathcal{R}(X)$ depends only on three variables. Then we have $n = r + 1$ and consequently $m = d - 1$. Further we may write T_i instead of T_{i1} and w_i instead of w_{i1} , etc.. In this setting, we obtain the following bounds for the numbers of possible varieties X (Fano or not).

Proposition 3.1. *For any pair $(d, \mu) \in \mathbb{Z}_{>0}^2$ there is, up to deformation, only a finite number of complete d -dimensional varieties with Picard number one, Picard index $[\text{Cl}(X) : \text{Pic}(X)] = \mu$ and Cox ring*

$$\mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_m] / \langle \alpha_i T_i^{l_i} + \alpha_{i+1} T_{i+1}^{l_{i+1}} + \alpha_{i+2} T_{i+2}^{l_{i+2}}; 0 \leq i \leq r-2 \rangle.$$

In this situation we have $r < \mu + \xi(\mu) - 1$ where $\xi(\mu)$ denotes the number of primes smaller than μ . Moreover for $w_i^0 \in \mathbb{Z}_{>0}$ and $u_k^0 \in \mathbb{Z}_{>0}$ where $0 \leq i \leq r$, $1 \leq k \leq m$, and the exponents l_i one has

$$l_i \leq \mu, \quad w_i^0 \leq \mu^r, \quad u_k^0 \leq \mu.$$

Proof. Consider the total coordinate space $\overline{X} \subseteq \mathbb{K}^{r+1+m}$ and the quotient $p: \widehat{X} \rightarrow X$ as well as the points $\overline{x}(k) \in \widehat{X}$ having the $(r+k)$ -th coordinate one and all others zero. Set $x(k) := p(\overline{x}(k))$. Then u_k^0 divides the order of the local class group $\text{Cl}(X, x(k))$. In particular we have $u_k^0 \leq \mu$.

For each $0 \leq i \leq r$ fix a point $\overline{x}(i) = (\overline{x}_0, \dots, \overline{x}_r, 0, \dots, 0)$ in \widehat{X} such that $\overline{x}_i = 0$ and $\overline{x}_j \neq 0$ for $i \neq j$. Then we obtain

$$\gcd(w_j^0, j \neq i) \mid |\text{Cl}(X, x(i))| \mid \mu.$$

By Lemma 2.6 we have $\text{lcm}_{j \neq i}(\gcd(l_i, l_j)) \mid \mu$. Now consider l'_i such that $l_i = \text{lcm}_{j \neq i}(\gcd(l_i, l_j)) \cdot l'_i$. Then the homogeneity condition $l_i w_i^0 = l_j w_j^0$ gives $l'_i \mid w_j^0$ for all $j \neq i$ and consequently $l'_i \mid \gcd(w_j^0, j \neq i)$ and $l'_i \leq \mu$. Since $l_i = l'_i \cdot \text{lcm}_{j \neq i}(\gcd(l_i, l_j))$ we can conclude $l_i \leq \mu$ by using Proposition 2.2. Since the l'_i are pairwise coprime we obtain $l'_0 \cdots l'_r \mid \gamma^0$ and $l'_0 \cdots l'_r \mid \mu$ where $\gamma^0 := \deg(g_0)^0 = l_i w_i^0$. From $l_i w_i^0 = l_j w_j^0$ we can follow

$$l_i = l_0 \frac{w_0^0}{w_i^0} = l_0 \frac{w_0^0 \cdots w_{i-1}^0}{w_1^0 \cdots w_{i-1}^0} = \eta_i \cdot \frac{\gcd(w_0^0, \dots, w_{i-1}^0)}{\gcd(w_0^0, \dots, w_i^0)} \leq \mu$$

where $1 \leq \eta_i \leq \mu$. In particular the last fraction is smaller than μ . All in all this gives us

$$\begin{aligned} w_0^0 &= \frac{w_0^0}{\gcd(w_0^0, w_1^0)} \cdot \frac{\gcd(w_0^0, w_1^0)}{\gcd(w_0^0, w_1^0, w_2^0)} \cdots \frac{\gcd(w_0^0, \dots, w_{r-2}^0)}{\gcd(w_0^0, \dots, w_{r-1}^0)} \cdot \gcd(w_0^0, \dots, w_{r-1}^0) \\ &\leq \mu^{r-1} \cdot \mu = \mu^r. \end{aligned}$$

Analogously we get the boundedness for all w_i^0 . Now let q be the number of l'_i that are greater than one. Since all l'_i , $0 \leq i \leq r$, are coprime q is bounded by $\xi(\mu)$ the number of primes smaller than μ . To avoid the toric case we assume $l_i \neq 1$ for all $0 \leq i \leq r$. Consequently if $l'_i = 1$ then there is at least one $0 \leq j \leq r$ such that $\gcd(l_i, l_j) > 1$. Since $\gcd(l_i, l_j)$ divides μ we get $r+1-q < \mu$ as a rough bound. All in all we get $r+1 = r+1-q+q < \mu + \xi(\mu)$. \square

Proof of Theorem 1.5. Let X be a variety as in Theorem 1.5. Then each monomial of the Cox ring relations depends on only one variable, i.e. $n_i = 1$ for $0 \leq i \leq r$. Consequently Proposition 3.1 provides bounds for the discrete data such as the non torsion parts of the weights w_{ij}^0 and u_k^0 , the exponents l_{ij} and the number of Cox ring relations r . Since $|\text{Cl}(X)^t| \leq \mu$ holds, the number of possibilities for the torsion part of the grading is also restricted which implies the assertion. \square

Lemma 3.2. Consider the ring $\mathbb{K}[T_{ij}; 0 \leq i \leq 2, 1 \leq j \leq n_i][S_1, \dots, S_k]/\langle g \rangle$ where $n_0 \geq n_1 \geq n_2 \geq 1$ holds and let K be a finitely generated abelian group of the form $K = \mathbb{Z} \oplus K^t$ with torsion part K^t . Suppose that g is homogeneous with respect to a K -grading of $\mathbb{K}[T_{ij}, S_k]$ given by $\deg T_{ij} =: w_{ij} = w_{ij}^0 + w_{ij}^t \in K$ with $w_{ij}^0 \in \mathbb{Z}_{>0}$ and $\deg S_k =: u_k = u_k^0 + u_k^t \in K$ with $u_k^0 \in \mathbb{Z}_{>0}$, and assume

$$\deg(g)^0 < \sum_{i=0}^2 \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^m u_i^0.$$

Let $\mu \in \mathbb{Z}_{>1}$, assume $w_{ij}^0 \leq \mu$ whenever $n_i > 1$, $1 \leq j \leq n_i$ and $u_k^0 \leq \mu$ for $1 \leq k \leq m$ and set $d := n_0 + n_1 + n_2 + m - 2$. Depending on the shape of g , one obtains the following bounds.

- (i) Suppose that $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} + \eta_2 T_{21}^{l_{21}}$ with $n_0 > 1$ and coefficients $\eta_i \in \mathbb{K}^*$ holds. If we have $l_{11} > l_{21} \geq 2$ then, one has

$$w_{11}^0 < 2d\mu, \quad w_{21}^0 < 3d\mu, \quad l_{22}, l_{21}, \deg(g)^0 < 6d\mu.$$

If we have $l_{11} = l_{21} \geq 2$ then, one has

$$l_{11}, w_{11}^0, l_{21}, w_{21}^0, \deg(g)^0 \leq \mu.$$

- (ii) Suppose that $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + \eta_2 T_{21}^{l_{21}}$ with $n_1 > 1$ and coefficients $\eta_i \in \mathbb{K}^*$ holds and we have $l_{21} \geq 2$. Then one has

$$w_{21}^0 < (d+1)\mu, \quad \deg(g)^0 < 2(d+1)\mu.$$

Proof. We prove (i). Set for short $c := (n_0 + m)\mu = d\mu$. Then, using homogeneity of g and the assumed inequality, we obtain

$$l_{11}w_{11}^0 = l_{21}w_{21}^0 = \deg(g)^0 < \sum_{i=0}^2 \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^m u_i^0 \leq c + w_{11}^0 + w_{21}^0.$$

First have a look at the case $l_{11} > l_{21} \geq 2$. Plugging this into the above inequalities, we arrive at $2w_{11}^0 < c + w_{21}^0$ and $w_{21}^0 < c + w_{11}^0$. We conclude $w_{11}^0 < 2c$ and $w_{21}^0 < 3c$. Consequently we obtain

$$\deg(g)^0 < c + w_{11}^0 + w_{21}^0 < 6c = 6d\mu.$$

If we have $l_{11} = l_{21}$ the homogeneity condition $l_{11}w_{11}^0 = l_{21}w_{21}^0$ gives us $w_{11}^0 = w_{21}^0$. Thus we have $\gcd(l_{11}, l_{21}) = l_{21} = l_{11} \mid \mu$ and $\gcd(w_{11}^0, w_{21}^0) = w_{11}^0 = w_{21}^0 \mid \mu$. Consequently $l_{11}, w_{11}^0, l_{21}, w_{21}^0, \deg(g)^0 \leq \mu$. We prove (ii). Here we set $c := (n_0 + n_1 + m)\mu = (d+1)\mu$. Then the assumed inequality gives

$$l_{21}w_{21}^0 = \deg(g)^0 < \sum_{i=0}^1 \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^m u_i^0 + w_{21}^0 \leq c + w_{21}^0.$$

Since we assumed $l_{21} \geq 2$, we can conclude $w_{21}^0 < c$. This in turn gives us $\deg(g)^0 < 2c$. \square

Theorem 3.3. *In the situation of Construction 2.1, fix the dimension $d = \dim(X)$ and the Picard index $\mu = [\text{Cl}(X) : \text{Pic}(X)]$ and let $\xi(x)$ denote the number of primes smaller than x . Then we have*

$$u_k^0 \leq \mu \quad \text{for } 1 \leq k \leq m \quad \text{and} \quad |\text{Cl}(X)^t| \leq \mu.$$

Moreover, for the free part of the degree of the relations γ^0 , the weights w_{ij}^0 and the exponents l_{ij}^0 , where $0 \leq i \leq r$ and $1 \leq j \leq n_i$ one obtains the following.

- (i) Suppose that $r = 0, 1$ holds. Then $n + m \leq d + 1$ holds and one has the bounds

$$w_{ij}^0 \leq \mu \quad \text{for } 0 \leq i \leq r \text{ and } 1 \leq j \leq n_i,$$

and the Picard index is given by

$$\mu = \text{lcm}(w_{ij}^0, u_k^0; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m) \cdot |\text{Cl}(X)^t|.$$

- (ii) Suppose that $r \geq 2$ and $n_0 = 1$ hold. Then $r \leq \mu + \xi(\mu) - 1$ and $n = r + 1$ and $m = d - 1$ hold and one has

$$w_{i1}^0 \leq \mu^r, \quad l_{i1} \mid \mu \quad \text{for } 0 \leq i \leq r, \quad \gamma^0 \leq \mu^{r+1},$$

and the Picard index is given by

$$\mu = \text{lcm}(\gcd_i(w_{j1}^0; i \neq j), u_k^0; 0 \leq i \leq r, 1 \leq k \leq m) \cdot |\text{Cl}(X)^t|.$$

- (iii) Suppose that $r \geq 2$ and $n_0 > n_1 = 1$ hold. Then we may assume $l_{11} \geq \dots \geq l_{r1} \geq 2$, we have $r \leq \mu + \xi(6d\mu) - 1$ and $n_0 + m = d$ and the bounds

$$w_{01}^0, \dots, w_{0n_0}^0 \leq \mu, \quad l_{01}, \dots, l_{0n_0} \leq 6d\mu, \quad \gamma^0 < 6d\mu,$$

$$w_{11}^0 < 2d\mu, \quad w_{21}^0 < 3d\mu, \quad w_{i1}^0, l_{i1} < 6d\mu \quad \text{for } 1 \leq i \leq r,$$

and the Picard index is given by

$$\mu = \text{lcm}(w_{0j}^0, \text{gcd}(w_{11}^0, \dots, w_{r1}^0), u_k^0; 1 \leq j \leq n_0, 1 \leq k \leq m) \cdot |\text{Cl}(X)^t|.$$

- (iv) Suppose that $n_1 > n_2 = 1$ holds. Then we may assume $l_{21} \geq \dots \geq l_{r1} \geq 2$, we have $r \leq \mu + \xi(2(d+1)\mu) - 1$ and $n_0 + n_1 + m = d+1$ and the bounds

$$w_{ij}^0 \leq \mu \quad \text{for } i = 0, 1 \text{ and } 1 \leq j \leq n_i, \quad w_{21}^0 < (d+1)\mu,$$

$$\gamma^0, w_{ij}^0, l_{ij} < 2(d+1)\mu \quad \text{for } 0 \leq i \leq r, \text{ and } 1 \leq j \leq n_i,$$

and the Picard index is given by

$$\mu = \text{lcm}(w_{ij}^0, u_k^0; 0 \leq i \leq 1, 1 \leq j \leq n_i, 1 \leq k \leq m) \cdot |\text{Cl}(X)^t|.$$

- (v) Suppose that $n_2 > 1$ holds and let s be the maximal number with $n_s > 1$. Then one may assume $l_{s+1,1} \geq \dots \geq l_{r1} \geq 2$, we have $s \leq d$, $r \leq \mu + \xi((d+2)\mu) + d - 1$ and $n_0 + \dots + n_s + m = d + s$ and the bounds

$$w_{ij}^0 \leq \mu, \quad \text{for } 0 \leq i \leq s, \quad \gamma^0 < (d+2)\mu,$$

$$w_{ij}^0, l_{ij} < (d+2)\mu \quad \text{for } 0 \leq i \leq r \text{ and } 1 \leq j \leq n_i,$$

and the Picard index is given by

$$\mu = \text{lcm}(w_{ij}^0, u_k^0; 0 \leq i \leq s, 1 \leq j \leq n_i, 1 \leq k \leq m) \cdot |\text{Cl}(X)^t|.$$

Proof. As before, we denote by $\overline{X} \subseteq \mathbb{K}^{n+m}$ the total coordinate space and we consider the quotient $p: \widehat{X} \rightarrow X$.

We first discuss the case that X is a toric variety. Then the Cox ring is a polynomial ring, $\mathcal{R}(X) = \mathbb{K}[S_1, \dots, S_m]$. For each $1 \leq k \leq m$, consider the point $\overline{x}(k) \in \widehat{X}$ having the k -th coordinate one and all others zero and set $x(k) := p(\overline{x}(k))$. Then, by Proposition 2.2, the order of the local class group $\text{Cl}(X, x(k))$ is divided by u_k^0 . Together with Corollary 2.4 we get $u_k^0 \leq \mu$ for $1 \leq k \leq m$ and $|\text{Cl}(X)^t| \leq \mu$ which settles Assertion (i).

Now we treat the non-toric case, which means $r \geq 2$. Note that we have $n \geq 3$. The case $n_0 = 1$ is done in Proposition 3.1. So, we are left with $n_0 > 1$. For every i with $n_i > 1$ and every $1 \leq j \leq n_i$, there is the point $\overline{x}(i, j) \in \widehat{X}$ with ij -coordinate T_{ij} equal to one and all others equal to zero, and thus we have the point $x(i, j) := p(\overline{x}(i, j)) \in X$. Moreover, for every $1 \leq k \leq m$, we have the point $\overline{x}(k) \in \overline{X}$ having the k -coordinate S_k equal to one and all others zero; we set $x(k) := p(\overline{x}(k))$. Proposition 2.2 provides the bounds

$$w_{ij}^0 \leq \mu, \quad u_k^0 \leq \mu \quad \text{for } 0 \leq i \leq r, 1 \leq j \leq n_i, n_i > 1, 1 \leq k \leq m.$$

Let $0 \leq s \leq r$ be the maximal number with $n_s > 1$. Then g_{s-2} is the last polynomial such that each of its three monomials depends on more than one variable. For any $t \geq s$, we have the “cut ring”

$$R_t := \mathbb{K}[T_{ij}, S_k] / \langle g_0, \dots, g_{t-2} \rangle$$

where $0 \leq i \leq t$, $1 \leq j \leq n_i$, $1 \leq k \leq m$ and the relations g_i depend on only three variables as soon as $i > s$ holds. For the free part of the degree γ^0 of the relations

we have

$$\begin{aligned}
(r-1)\gamma^0 &= (t-1)\gamma^0 + (r-t)\gamma^0 \\
&= (t-1)\gamma^0 + l_{t+1,1}w_{t+1,1}^0 + \dots + l_{r1}w_{r1}^0 \\
&< \sum_{i=0}^r \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^m u_i^0 \\
&= \sum_{i=0}^t \sum_{j=1}^{n_i} w_{ij}^0 + w_{t+1,1}^0 + \dots + w_{r1}^0 + \sum_{i=1}^m u_i^0.
\end{aligned}$$

Since $l_{i1}w_{i1}^0 > w_{i1}^0$ holds in particular for $t+1 \leq i \leq r$, we derive from this the inequality

$$\gamma^0 < \frac{1}{t-1} \left(\sum_{i=0}^t \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^m u_i^0 \right).$$

To obtain the bounds in Assertions (iii) and (iv), we consider the cut ring R_t with $t = 2$ and apply Lemma 3.2 and Proposition 2.2; note that we have $d = n_0 + n_1 + n_2 + m - 2$ for the dimension $d = \dim(X)$ and that $l_{21} \geq 2$ is due to the fact that X is non-toric. The bounds $w_{i1}^0, l_{i1} < 6d\mu$ for $3 \leq i \leq r$ in Assertion (iii) follow from $\gamma^0 < 6d\mu$. Similarly $w_{ij}^0, l_{ij} < 2(d+1)\mu$ for $0 \leq i \leq r, 1 \leq j \leq n_i$ in Assertion (iv) follow from $\gamma^0 < 2(d+1)\mu$. We still have to proof the restriction for the number of relations, which means bounding r . Set $\ell_i = \text{lcm}_{0 \leq j \neq i \leq r}(\gcd(\ell_i, \ell_j)) \cdot \ell'_i$. Then ℓ'_0, \dots, ℓ'_r are coprime. For $i \geq 1$ we have $n_i = 1$. Thus analogously to the proof of Proposition 3.1 we get $r+1 = r+1-q+q \leq \mu + \xi(6d\mu)$ where q is the number of ℓ'_i that are greater than one and satisfy $n_i = 1$. For the bound in assertion (iv) the same argument yields $r+1 = r+1-q+q \leq \mu + \xi(2(d+1)\mu)$.

To obtain the bounds in Assertion (v), we consider the cut ring R_t with $t = s$. Using $n_i = 1$ for $i \geq t+1$, we can estimate the degree of the relation as follows:

$$\gamma^0 \leq \frac{(n_0 + \dots + n_t + m)\mu}{t-1} = \frac{(d+t)\mu}{t-1} \leq (d+2)\mu.$$

We have $w_{ij}^0 l_{ij} \leq \gamma^0$ for any $0 \leq i \leq r$ and any $1 \leq j \leq n_i$, which implies that all w_{ij}^0 and l_{ij} are bounded by $(d+2)\mu$. Since $n_0, \dots, n_{s-1} > 1$ holds, the number s is bounded by $s = 2s - (s-1) - 1 \leq d$. Consequently we obtain $r+1 = r+1-s-q+s+q \leq \mu + \xi((d+2)\mu) + d$, where q is defined as above.

Finally, we have to express the Picard index μ in terms of the free part of the grading weights w_{ij}^0, u_k^0 and the torsion part $\text{Cl}(X)^t$ as claimed in the assertions. This is a direct application of the formula of Proposition 2.2. \square

Proof of Theorem 1.1. Theorem 3.3 provides bounds for the exponents and the number of relations as well as for the free part of the grading weights and the torsion part of $\text{Cl}(X)$. Since we have $|\text{Cl}(X)^t| \leq \mu$ the possibilities for the torsion part of the grading weights are also restricted. Suitably estimating yields that the number $\delta(d, \mu)$ of different deformation types is bounded by

$$\mu^{\mu^2 + 3\mu + \xi(\mu)^2 + \xi(6d\mu) + 5d} (6d\mu)^{2\mu + 2\xi(6d\mu) + 3d - 2}$$

which leads to the results of Theorem 1.1. \square

Proof of Theorems 1.2 and 1.4. For fixed d and μ Theorem 3.3 bounds the number of possible data l_{ij}, w_{ij}^0, u_k^0 , belonging to Fano varieties. We identify all these constellations by a computer based algorithm. Since $|\text{Cl}(X)^t| \leq \mu$ holds, there is only a finite number of possibilities for the torsion part of the grading weights that we have to run through. By this procedure we receive the tables of 1.2 and 1.4.

We claim that any two of the listed Cox rings do not describe varieties that are isomorphic to each other. Two minimal systems of homogeneous generators of the Cox ring contain (up to reordering) the same free parts of generator degrees w_{ij}^0 , $u_k^0 \in \mathbb{Z}$. Consequently they are invariant under isomorphism. Further the exponents $l_{ij} > 1$ are representing the orders of all finite non-trivial isotropy groups of one-codimensional orbits of the action T on X ; see [7, Theorem 1.3]. Moreover, since none of the listed Cox rings is polynomial the varieties are all non-toric. This implies that every complexity one action can be assigned to a maximal torus in $\text{Aut}(X)$. Since the maximal tori are all conjugated the varieties with complexity one torus action are isomorphic if and only if they are T -equivariantly isomorphic. Thus, running through the exponents l_{ij} we receive that any two of the varieties listed in Theorem 1.2 are not isomorphic.

In case of Theorem 1.4 there is some more work to do. There are not isomorphic threefolds varying only in the torsion part of the grading weights, see for example number 2, 3 and 4. In these cases, comparing the torsion parts of the gradings shows that it is not possible to install a $\text{Cl}(X)$ -graded ring isomorphism between the Cox rings of two different threefolds.

We consider exemplarily the threefolds number 2 and 3: Let D_2 be a prime divisor, representing $\deg(T_2) \in \text{Cl}(X)$ and let E_1 be a prime divisor, representing $\deg(S_1) \in \text{Cl}(X)$. Then D_2 has isotropy group of order $l_2 = 3$ and E_1 has infinite isotropy. In case of threefold number 2 the term $D_2 - E_1$ represents a non-trivial torsion element where as in case of threefold number 3 it is the zero element in $\text{Cl}(X)$. Thus, these two varieties are not isomorphic. Analogously we proceed with all other cases to obtain finally the list of Theorem 1.4. \square

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